MATHEMATICAL CROCHET: BOY'S SURFACE

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Abstract. In this paper, we describe how to crochet Boy's surface and give an introductory overview of the mathematics behind it: the real projective plane $\mathbb{RP}^2$, and how Boy's surface lets us picture it inside ordinary 3-dimensional space.

The total time to crochet this Boy's surface is roughly four hours.

(a) A crocheted Boy's surface, Gus (the cat) is not in the scope of this paper.

(b) The steel Boy's surface sculpture outside the library of the Mathematisches Forschungsinstitut Oberwolfach.

Figure 1. Two realizations of Boy's surface: a crocheted model (left) and a large steel sculpture (right).

1. Mathematical Background

1.1. **The real projective plane.** The **real projective plane**, written $\mathbb{RP}^2$, is a two-dimensional manifold: a shape that looks flat if you zoom in far enough, like a sphere or a Klein bottle. It can be built by taking a disk (think of a flat, circular piece of fabric) and gluing every point on its boundary circle to the point directly opposite it on that same circle (its **antipodal** point). So instead of leaving the disk's edge alone, we fold the boundary circle onto itself, matching up antipodal pairs of points two at a time.

Just like the Klein bottle, $\mathbb{RP}^2$ is **non-orientable**: there is no consistent way to define an inside and an outside, and no consistent way to define a "clockwise" direction of rotation everywhere on the surface. If you drew a clock face at one point on $\mathbb{RP}^2$ and slid it around a suitable loop, by the time it came back to where it started it would be running counterclockwise.

1.2. **A video-game analogy.** A helpful way to build intuition for these gluing rules is to think about how the classic video game *Snake* handles the edges of the screen. In the most common version of *Snake*, when the snake moves off the right edge of the screen, it reappears at the *same height* on the left edge, and similarly for the top and bottom edges. Gluing the left and right edges of the screen together directly (and, separately, the top and bottom edges directly) is exactly the square-with-identified-edges picture of a **torus** (the surface of a donut).

$\mathbb{RP}^2$ corresponds to a different, stranger version of *Snake*. Imagine that when the snake moves off the right edge of the screen, it reappears on the *left* edge, but at the *mirrored* height (so exiting near the top of the right edge brings you back in near the bottom of the left edge). The same mirrored wraparound happens at the top and bottom edges. This "flip on every wraparound" rule is precisely the antipodal identification described above, now applied to a square instead of a disk: it identifies each point on the boundary with the point reached by flipping across the center. Figure 2 shows this square picture of $\mathbb{RP}^2$: unlike a torus (both pairs of edges glued straight across) or a **Möbius strip** (a strip of paper given a half twist and taped end to end, which corresponds to flipping just one pair of edges), *both* pairs of edges here are glued with a flip.

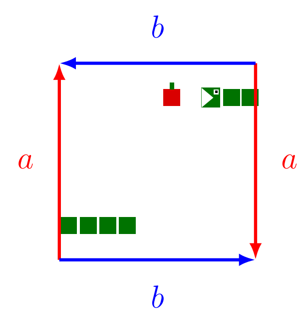


Figure 2. The real projective plane $\mathbb{RP}^2$ as a square: both the red edges and the blue edges are identified with a twist (opposite-direction arrows). There is no free boundary left over; every edge of the square is glued to another. The snake illustrates the resulting wraparound rule: heading west, it leaves through the red edge low on the left, and reappears through the red edge high on the right, still heading west, in reach of the apple. To play *Snake* on $\mathbb{RP}^2$, and to watch the corresponding point move on Boy's surface, click here.

1.3. **Boy's surface.** Unfortunately, just like the Klein bottle, $\mathbb{RP}^2$ cannot be built in ordinary 3-dimensional space without the surface passing through itself. Werner Boy discovered in 1901 that $\mathbb{RP}^2$ *can* be immersed in 3-dimensional space (that is, drawn in 3-dimensional space allowing self-intersection, but with no sharp creases or pinch points); this immersed surface is called **Boy's surface**, and it is the shape we crochet in this paper.

The discovery was something of a surprise, including to Boy himself. His doctoral advisor, David Hilbert, had set him the task of *proving* that no such surface could exist [Boy03; Wik]. Two pictures of $\mathbb{RP}^2$ inside 3-dimensional space were already known, called the **cross-cap** and the **Roman surface**, but both have bad points where the surface pinches to a sharp corner, and Hilbert expected that any attempt would run into the same problem. Instead of the impossibility proof he was asked for, Boy produced a counterexample: a surface that only passes through itself, smoothly, with no pinch points anywhere [Wik].

As with the Klein bottle, this self-intersection is unavoidable: $\mathbb{RP}^2$ is not **embeddable** in 3-dimensional space (that is, it cannot be realized there without self-intersection), although, like the Klein bottle, it *can* be embedded in 4-dimensional space.

Boy's surface also turns up in the study of **sphere eversion**: the fact, proved by Stephen Smale in 1958, that a sphere can be turned inside out continuously, without ever tearing or creasing it, provided the surface is allowed to pass through itself along the way [Sma58]. Working out how to actually picture such a process took mathematicians decades longer, and in some of the resulting animations, such as *Outside In* [LMM94], the sphere passes through a shape built out of Boy's surface partway through.

Boy's surface is also something of a landmark within the mathematical community: a large steel model of it, larger than a person, stands outside the library of the Mathematisches Forschungsinstitut Oberwolfach (MFO) in Germany, installed in 1991 as a gift from Mercedes-Benz [Mat91]. Oberwolfach hosts a constant stream of visiting mathematicians for its week-long workshops, and its Boy's surface sculpture has become a favorite backdrop for their portraits, so that many well-known mathematicians have had their portraits taken in front of it over the years (Figure 1).

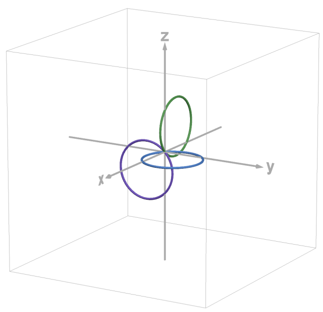
2. Crochet Instructions

2.1. **The short version.** Make three loops, attach them all at a single point, then extend each loop via a spiral, connecting it to one of the other loops at every round.

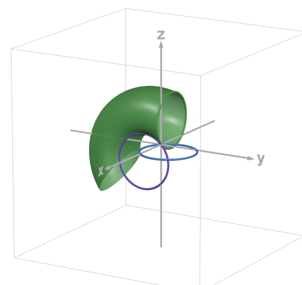
That is the entire construction; the rest of this section fills in the details. Figure 3 shows the same three loops as a parametric model in $\mathbb{R}^3$ (the equations are given in Appendix B). An interactive version is available at

https://www.desmos.com/3d/hkix136yvm

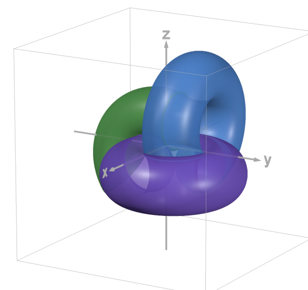
and we recommend manipulating it directly: rotating the plot and sweeping the loops out one at a time makes the three-fold structure much easier to see than any static picture.



(a) The three starting loops, each pair meeting at the single shared point.



(b) The green loop (Loop 2) partially swept, using the purple loop (Loop 3) as its spine.



(c) All three loops fully swept.

Figure 3. A parametric model of the three interlocking loops, built in Desmos 3D.

2.2. **General overview.** Boy's surface has a natural three-fold symmetry (it is often pictured with three lobes meeting at a single point), and we exploit this symmetry in the construction: we begin with three separate pieces, labeled **Loop 1**, **Loop 2**, and **Loop 3** (each crocheted from its own ball of yarn; different colors are recommended, so the three loops remain easy to tell apart – in our photos, Loop 1 is purple, Loop 2 is orange, and Loop 3 is blue; see Figure 4). The setup is worked the same way on all three: chain 20 and join into a ring with a slip stitch (Table 1). Loop 2 and Loop 3 are then fastened off and set aside, each with a long tail left over.[1] Loop 1 is kept on the hook and continues growing.

Before growing further, though, Loop 1 is joined to the other two at a single shared point: the hook is inserted through the base ring of *both* Loop 2 and Loop 3 at once, and a slip stitch is pulled through all of them together. This single point, where all three starting rings meet, will sit at the heart of Boy's surface's three-fold symmetry; the rest of the construction builds outward from it.

[1]Cutting the yarn is not strictly necessary: one could leave all three loops attached to their balls of yarn throughout. In practice, though, three live strands get badly tangled as the piece grows, so it is much easier to cut, and the tails are easy to hide inside the surface at the end.

Loop 1 is then worked in continuous spiral rounds: no chain and no joining slip stitch between rounds, just sc directly into the next stitch, round after round. Each round also attaches one stitch of Loop 2, which acts as Loop 1's **spine**: after 19 sc, a slip stitch is worked into the next stitch of Loop 2, attaching it; then the next stitch of Loop 1 is *skipped* and a final sc is made, so that Loop 1 stays at 20 stitches per round despite the extra join. This same round is repeated 17 times (Rows A1–A17 of Table 2; Figure 5). Loop 1 is then left live, without fastening off. Note that each spine ring has 20 stitches but only 17 rounds are worked, so three stitches of the spine are left unattached; this is fine, as they end up covered when the edges are closed off at the finish (Row D2 of Table 5).

The same process is then repeated one loop over: a live loop is added to Loop 2, which was fastened off earlier (see §2.5 for how), and Loop 2 is worked in continuous spiral rounds exactly as Loop 1 was, this time using Loop 3 as *its* spine (Rows B1–B17 of Table 3; Figure 6). Loop 2 is then left live, without fastening off.

Finally, the pattern repeats a third time, closing the three-fold symmetry: a live loop is added to Loop 3, which is then worked in continuous spiral rounds using Loop 1 as its spine (Rows C1–C17 of Table 4). This closes the cycle Loop 1 → Loop 2 → Loop 3 → Loop 1 back at the single shared point from the initial join: by the end of Table 4, the three live loops left open along the way (Loop 1's, from the end of Table 2; Loop 2's, from the end of Table 3; and Loop 3's, currently on the hook) have all been brought back together at that same point. There is an important subtlety here, discussed in §2.5: the instructions do not say which way *around* Loop 1's base to travel, and only one of the two directions preserves the three-fold symmetry and brings the three live edges together.

With the three live loops together, the piece is stuffed (Row D1 of Table 5), adding the stuffing gradually and evenly as the piece fills, rather than all at once, to avoid lumps and get an even fill (Figure 8).

The three loops are then stitched together, closing the piece with a **decreasing spiral** shared between all three live knots at once. First, with the three live loops together, their three open edges are closed to one another with slip stitches (Row D2; see §2.5), bringing the total down to 30 stitches spread across three live knots, 10 stitches at each knot. From there, each knot is decreased symmetrically and in step with the other two: from 10 down to 8 stitches (Row D3: sc, sc, dcr, sc, sc, dcr, sc, sc), then to 6 (Row D4: sc, dcr, sc, sc, dcr, sc), then to 4 (Row D5: sc, dcr, sc, dcr), and finally to 3 (Row D6: sc, dcr, sc), leaving 9 stitches in total across the three knots. At that point the yarn is cut, leaving a tail, and a yarn needle is used to thread the tail through all 9 remaining stitches and pull the opening closed (Figure 9). Finally, the piece is massaged and given more stuffing if needed, to settle into a nice, even shape.

2.3. Glossary of abbreviations.

(1) ch = chain.
(2) sl st = slip stitch.
(3) inc = increase (work two single crochets into one stitch).
(4) sc = single crochet.
(5) dcr = single crochet decrease.
(6) fasten off = cut the yarn, leaving a tail, and pull the tail through the last loop to secure it.
(7) spine = the loop that the loop currently being worked is attached to, one stitch per round. Each of the three loops uses the next one as its spine, so that the three end up interlocked.
(8) live knot = one of the three still-open ends of the piece, each carrying its own live loop. After the three edges are closed together (Row D2 of Table 5), the three live knots are decreased in step with one another until the piece closes.

2.4. **Detailed instructions.** Unless noted otherwise, every round below is worked in a **continuous spiral**: no chain and no joining slip stitch between rounds (refer to *glossary of abbreviations* §2.3 and *steps requiring more explanation* §2.5 for more details; the **Sts** column gives the stitch count *after* each row). Rows are labeled by which piece they belong to: the A rows are Loop 1's, the B rows are Loop 2's, and the C rows are Loop 3's (Tables 2–4), while the D rows close up the finished piece (Table 5). Rows B0 and C0 simply put a loop back on the hook, so the actual crocheted rounds are A1–A17, B1–B17, and C1–C17. The initial setup (Table 1) is worked identically on all three loops before any of this begins.

Instructions	**Sts**
ch 20, sl st into first ch to form a ring	20
Loop 2 & 3: fasten off (long tail); Loop 1: continue	20 (Loop 1)
Loop 1: sl st through the base rings of Loop 2 & 3, joining all three at one point	20

Table 1. Initial setup: form all three base rings identically, then fasten off Loops 2 and 3 and join the three rings at one point.

Row	Instructions	Sts
A1 – A17	sc 19, sl st into next Loop 2 st (spine), skip 1 st, sc 1	20

Table 2. Loop 1, using Loop 2 as its spine (17 rounds). Loop 1 is left live afterward.

Row	Instructions	Sts
B0	add a live loop to Loop 2 (§2.5)	20
B1 – B17	sc 19, sl st into next Loop 3 st (spine), skip 1 st, sc 1	20

Table 3. Loop 2, using Loop 3 as its spine (17 rounds). Loop 2 is left live afterward.

Row	Instructions	Sts
C0	add a live loop to Loop 3 (§2.5)	20
C1 – C17	sc 19, sl st into next st of Loop 1's base ring (mind the direction, §2.5), skip 1 st, sc 1	20

Table 4. Loop 3, using Loop 1's base ring as its spine (17 rounds). This closes the three-fold cycle Loop 1 → Loop 2 → Loop 3 → Loop 1. Only one of the two directions around the base ring is correct (§2.5).

Row	Instructions	Sts
D1	stuff gradually & evenly, for a smooth fill	20 each (3 knots)
D2	close the 3 edges together (§2.5)	30 (10/knot)
D3	for each knot: sc, sc, dcr, sc, sc, dcr, sc, sc	8/knot (24 total)
D4	for each knot: sc, dcr, sc, sc, dcr, sc	6/knot (18 total)
D5	for each knot: sc, dcr, sc, dcr	4/knot (12 total)
D6	for each knot: sc, dcr, sc	3/knot (9 total)
D7	cut yarn; thread tail through all 9 sts, pull closed	0 (closed)

Table 5. Closing: the three live knots are stuffed and joined together in a decreasing spiral.

After closing, massage the piece and add more stuffing if needed, to get a nice, even shape.

2.5. **Steps requiring more explanation.**

(1) **"add a live loop" in Rows B0 and C0 of Tables 3 and 4.** We are not recovering the old live loop, but rather *adding a new one*: insert the hook into the stitch where you want to resume, draw up a loop of yarn, and continue as usual. In standard pattern language, this is joining (or attaching) new yarn to that stitch with a slip stitch.

(2) **Which way around Loop 1's base to work in Rows C1–C17 of Table 4.** Loop 3 is spiraled around Loop 1's base ring, exactly as Loop 1 was spiraled around Loop 2's and Loop 2 around Loop 3's. What is *not* determined by that instruction is the direction of travel: one can go around the base ring clockwise or counterclockwise, and only one of the two is correct.

The rule is that Loop 3 must go around in the direction that continues the pattern already set by the first two loops, so that the same handedness is used at all three joins. Choose that direction and the construction respects the three-fold symmetry: the three loops come back together with their live edges meeting at the shared point, ready to be closed off. Choose the other direction and the symmetry is broken; Loop 3's spiral finishes on the wrong side, and its live edge ends up nowhere near the other two, so the three cannot be joined.

Figure 7 shows how to check this before committing to it. The two black arrows mark the direction in which Loop 1's and Loop 2's spirals are traveling around their spines; the yellow arrow marks the direction Loop 3 must be started around Loop 1's base ring in order to agree with them.

(3) **"close the 3 edges together" in Row D2 of Table 5.** At this stage the three loops are all live and meet in the same place, each with an open edge of 20 stitches, for 60 stitches in total. Going around, work 5 slip stitches that join one loop's edge to its neighbor's, taking one stitch from each loop per slip stitch, and repeat for each of the three loops (15 joining slip stitches in all). This seams the three edges to one another and consumes 30 of the 60 stitches, leaving a single joined round of 30 live stitches, split into three live knots of 10 stitches each (Figure 9).

3. Further Reading

If you would like to learn more about Boy's surface and the topology behind it, here are some suggestions, roughly ordered from most to least accessible.

- To watch a sphere being turned inside out, see the computer-animated film *Outside In*, made at the Geometry Center by Silvio Levy, Delle Maxwell, and Tamara Munzner [LMM94]. It requires no background at all, and is the fastest way to get a feel for what it means for a surface to pass through itself without creasing.

- For an introduction that assumes no calculus, read Jeffrey Weeks' book *The Shape of Space* [Wee02]. It is written for a general audience, and introduces $\mathbb{RP}^2$, Möbius strips, and non-orientable surfaces through hands-on paper models, much like the crochet construction here.
- For a classic of mathematical exposition, read David Hilbert and Stephan Cohn-Vossen's book *Geometry and the Imagination* [HC52], by the same Hilbert who set Boy his thesis problem. It discusses the projective plane and non-orientable surfaces with an emphasis on pictures and intuition rather than formalism.
- To understand how surfaces like Boy's surface are drawn, and how their self-intersections fit together, read George Francis' book *A Topological Picturebook* [Fra87]. It is a good bridge between the popular accounts above and the textbooks below.
- For the precise definitions behind §1, read James Munkres' textbook *Topology* [Mun00]. This is the standard undergraduate reference for point-set topology and manifolds.
- To see $\mathbb{RP}^2$ placed among *all* closed surfaces, read William Massey's textbook *A Basic Course in Algebraic Topology* [Mas91]. It is a graduate text, and it classifies closed surfaces using polygon edge identifications, the same technique behind Figure 2.
- For orientability, immersions, and embeddings treated with full rigor, read Manfredo do Carmo's textbook *Differential Geometry of Curves and Surfaces* [Car76]. It is a classic upper-level undergraduate text.

Appendix A. Supplemental Images

(a) The three starting loops: Loop 1 (purple), Loop 2 (orange), Loop 3 (blue).

(b) The three loops joined at a single point.

Figure 4. The three loops before and after the initial join.

(a) Partway through the spiral, being slowly attached to Loop 2's spine.

(b) The spiral continued further.

Figure 5. Loop 1's spiral progressing, attaching to Loop 2's spine.

Figure 6. Loop 2 being worked with Loop 3 as its spine.

Figure 7. The critical step: deciding which way around Loop 1's base ring to work Loop 3. The black arrows show the direction of travel already established by Loop 1's and Loop 2's spirals; the yellow arrow shows the direction Loop 3 must be started in to agree with them. The other direction breaks the three-fold symmetry and leaves Loop 3's live edge on the wrong side.

(a) All three loops finished, joined at the single shared point.

(b) Stuffing being applied (Row D1).

Figure 8. Finishing the three loops, then stuffing.

(a) The first three edges joined (Row D2).

(b) The resulting round, three live knots at 10 sts each (Row D2).

(c) The result of the spiral decrease (Rows D3–D6).

Figure 9. Joining the three loops together and decreasing to close the piece.

Appendix B. A parametric model of the three loops

To make the three-fold construction concrete, we can model Loop 1, Loop 2, and Loop 3 (see §2.2) as three circles in $\mathbb{R}^3$, shown in blue, green, and purple respectively in Figure 3 (these illustrative colors are unrelated to the yarn colors used in the crochet photographs elsewhere in this paper):

$$\begin{aligned}
\text{Loop 1 (blue):}\quad & C_1(t) = \left(\sqrt{2}\cos t + 1,\ \sqrt{2}\sin t + 1,\ 0\right),\\
\text{Loop 2 (green):}\quad & C_2(t) = \left(\sqrt{2}\sin t - 1,\ 0,\ \sqrt{2}\cos t + 1\right),\\
\text{Loop 3 (purple):}\quad & C_3(t) = \left(0,\ \sqrt{2}\sin t - 1,\ -\sqrt{2}\cos t - 1\right).
\end{aligned}$$

Each pair of these circles meets at a single point, the origin.

To model each loop growing by spiraling around the next one as its spine (Tables 2–4), we sweep each circle by rigidly rotating it about the central axis of the *next* circle. Concretely, a circle centered at p in a plane with unit normal $\hat{n}$ is exactly the orbit of any one of its points under rotation about the line through p in direction $\hat{n}$; rotating a point $q \in \mathbb{R}^3$ by angle v about that line sweeps q once around the circle as v ranges over $[0, 2\pi]$. Applying this same rotation to *every* point of the other loop, rather than just to the shared intersection point, produces a surface $S(u, v)$: a one-parameter family of circles (parametrized by u) that each still touch the spine at exactly one point, but with that point of contact now sliding around the spine as the sweep parameter v increases – exactly the picture of a spiral round attaching to the next stitch of its spine (Figure 3, center and right panels). Carrying this out for each of the three pairs gives

$$S_{12}(u,v) = \Big((\sqrt{2}\cos u + 2)\cos v + \sin v - 1, \quad \sqrt{2}\sin u + 1, \quad (\sqrt{2}\cos u + 2)\sin v - \cos v + 1 \Big),$$

$$S_{23}(u,v) = \Big(\sqrt{2}\sin u - 1, \quad \cos v - (\sqrt{2}\cos u + 2)\sin v - 1, \quad \sin v + (\sqrt{2}\cos u + 2)\cos v - 1 \Big),$$

$$S_{31}(u,v) = \Big(1 - \cos v - (\sqrt{2}\sin u - 2)\sin v, \quad 1 - \sin v + (\sqrt{2}\sin u - 2)\cos v, \quad -\sqrt{2}\cos u - 1 \Big),$$

where S_{12} sweeps Loop 1 around Loop 2's spine, S_{23} sweeps Loop 2 around Loop 3's spine, and S_{31} sweeps Loop 3 around Loop 1's base ring, matching the cycle Loop 1 $\to$ Loop 2 $\to$ Loop 3 $\to$ Loop 1 described in §2.2. The three sweeps all run in the same rotational sense, which is the parametric counterpart of working all three loops around their spines the same way (§2.5).

References

[Boy03] W. Boy. "Über die Curvatura integra und die Topologie geschlossener Flächen". In: *Mathematische Annalen* 57.2 (1903), pp. 151–184.

[Car76] M. P. do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976.

[Fra87] G. K. Francis. *A Topological Picturebook*. Springer-Verlag, 1987.

[HC52] D. Hilbert and S. Cohn-Vossen. *Geometry and the Imagination*. Chelsea Publishing Company, 1952.

[LMM94] S. Levy, D. Maxwell, and T. Munzner. *Outside In*. 1994.

[Mas91] W. S. Massey. *A Basic Course in Algebraic Topology*. Vol. 127. Graduate Texts in Mathematics. Springer-Verlag, 1991.

[Mat91] Mathematisches Forschungsinstitut Oberwolfach. *The Boy Surface at Oberwolfach*. 1991. URL: https://www.mfo.de/about-the-institute/history/boy-surface.

[Mun00] J. R. Munkres. *Topology*. 2nd. Prentice Hall, 2000.

[Sma58] S. Smale. "A Classification of Immersions of the Two-Sphere". In: *Transactions of the American Mathematical Society* 90.2 (1958), pp. 281–290.

[Wee02] J. R. Weeks. *The Shape of Space*. 2nd. CRC Press, 2002.

[Wik] Wikipedia contributors. *Boy's Surface*. URL: https://en.wikipedia.org/wiki/Boy%27s_surface.

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