MATHEMATICAL CROCHET: KLEIN BOTTLE

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Abstract. In this paper, we describe how to crochet a Klein bottle and give an introductory overview of the mathematics behind the Klein bottle.

We give two constructions. The first builds the Klein bottle directly as a single spiraling piece and takes roughly one hour. The second builds it by sewing together two Möbius loops with opposite directions of half-twist and takes roughly two hours.

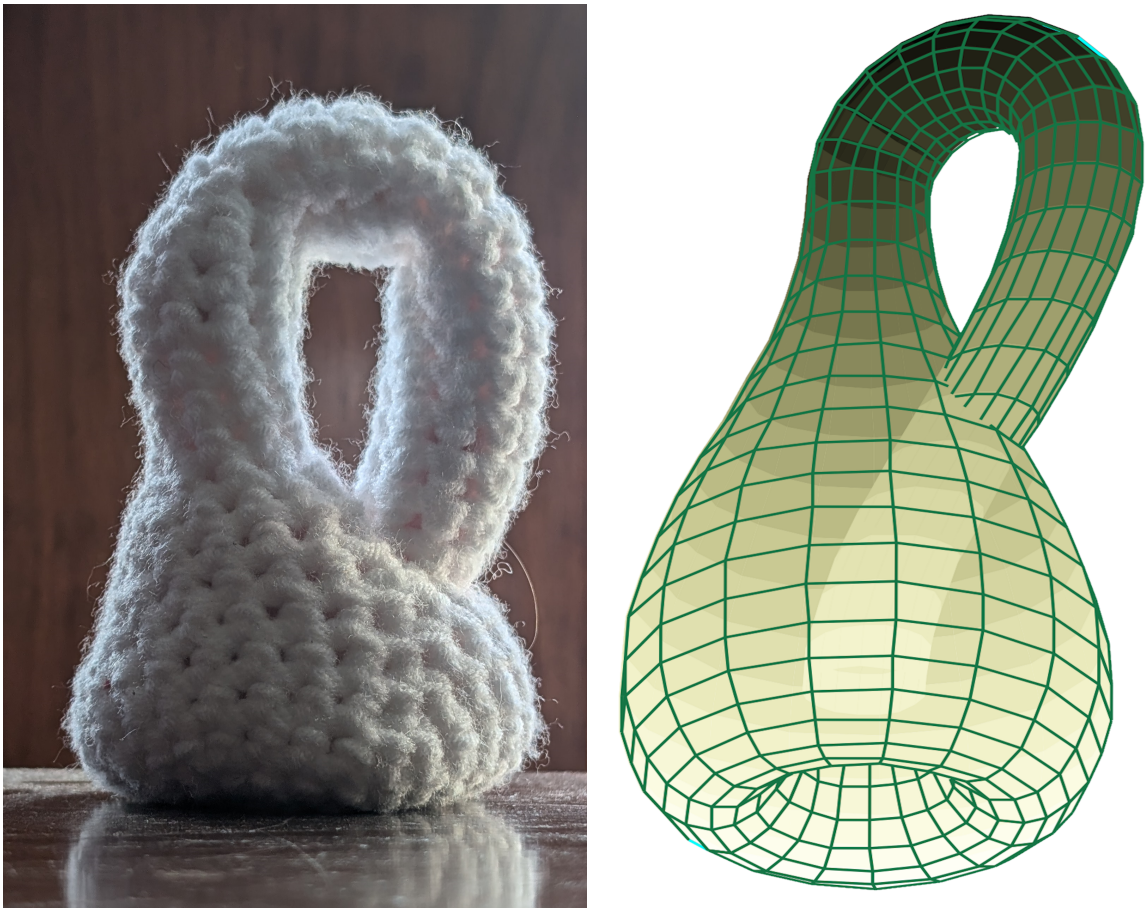


Figure 1. Left: A crocheted approximation of a Klein bottle. Right: A visualization of a Klein bottle, adapted from Wikipedia [Ttt06] and rotated to match the orientation of the crocheted model.

1. Mathematical Background

The Klein bottle is a mathematical surface discovered by Felix Klein in 1882. It is an example of a **non-orientable two-dimensional manifold**. A **two-dimensional manifold** is (essentially) any shape that looks flat if you zoom in far enough. For example, a sphere (like the earth) is a **two-dimensional manifold** because we can zoom in and it looks flat. A two-dimensional manifold is **orientable** if there is a consistent way of defining an inside and outside. For example, if you have a balloon with one ant inside and one ant outside, the two ants cannot reach each other. The balloon is an **orientable** two-dimensional manifold. The Klein bottle is **non-orientable**, there is no consistent way of defining an inside and an outside. No matter where you put two ants, they can always walk on the surface to meet each other.

One noticeable feature of the Klein bottle presented here is that it intersects itself. Namely, it has a portion where the neck goes into the base. This self-intersection will be present in *every* Klein bottle that exists in our 3-dimensional space. This is an example of a **immersed** but not **embedded** manifold. An **immersed** manifold allows intersection, while an **embedded** manifold does not allow intersection.

An interesting fact is that the Klein bottle can be **embedded** in 4-dimensional space (that is, it can be described in 4-dimensional space without having a self-intersection).

This is analogous to tying a knot in a string, and flattening it. Before we flatten it, the string does not self-intersect (that is, the string is a 1-dimensional embedded manifold in 3-dimensional space). But once we flatten it, the string becomes an immersed (but not embedded) 1-dimensional manifold on 2-dimensional space.

2. Crochet Instructions

2.1. Glossary of abbreviations.

(1) inc = increase (work two single crochets into one stitch).
(2) sc = single crochet.
(3) dcr = single crochet decrease.

2.2. **General overview.** The way the Klein bottle is crocheted is to begin at the base with a loop. We then spiral around the loop, increasing the circumference as we go. After five rounds, we begin decreasing the number of stitches. During this shrinking procedure, a hole is made to later pass the neck of Klein bottle through itself. Once the Klein bottle shrinks to the size of its neck, it is lengthened to roughly three times its height. The neck is then passed through the hole, the Klein bottle is filled with stuffing, and the neck is stitched to the loop constructed in the first step.

Row	Instructions	Number of Stitches
1	chain 12, slip last stitch into first	12
2	6×(inc, sc)	18
3	6×(inc, 2 sc)	24
4	6×(inc, 3 sc)	30
5	30×sc	30
6	3×(dcr, 8 sc)	27
7	3×(dcr, 7 sc)	24
8	3×(dcr, 6 sc)	21
9	make hole	18
10	3×(dcr, 4 sc)	15
11	3×(dcr, 3 sc)	12
12	3×(dcr, 2 sc)	9
13–26	9 sc	9
27	3×(inc, 2 sc)	12
28–29	12 sc	12
30	fill and connect to base	12

Table 1. Row by row instructions. The first column is the row number. The second column is the instructions (refer to *glossary of abbreviations* §2.1 and *steps requiring more explanation* §2.3 for more details). The third column is the number of stitches in the row *after* completing the instructions in the second column.

2.3. **Steps requiring more explanation.** There are two steps in the instructions that require more detail than would fit in the table.

(1) **"make hole" in the 9th row.** Here we decrease, then do one single crochet. Then we chain three, and slip stitch into the fourth stitch from where the chain started. This creates a loop with a circumference of six stitches. Then we single crochet. Then we decrease, do five single crochet stitches, decrease, then do another five single crochet stitches. This completes the 9th row.

(2) **"fill and connect to base" in the 30th row.** Here we fill the Klein bottle with stuffing[1]. The crochet hook can be used to stuff the neck. We then pass the neck through the hole created in the 9th row. We must cut the yarn. Make sure to have at least 12 stitches of yarn remaining. We then slip stitch into the original 12 stitches of the loop.

[1]It is important to note that we could fill the bottle *after* connecting the neck to the base. This is an important feature of the Klein bottle. However, working the stuffing through the neck and inside the bottle would be rather time-consuming.

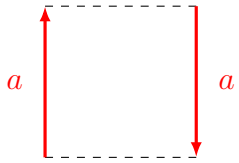
3. Crochet Instructions for Alternative Construction

In this section, we construct a Klein bottle by sewing together two Möbius loops with opposite directions of half-twist. The approximate time to construct this Klein bottle is two hours.

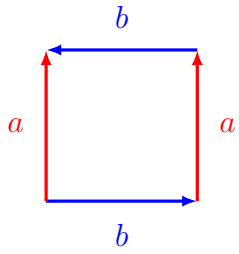
3.1. From Möbius strips to the Klein bottle.

This construction builds the Klein bottle out of two **Möbius strips**. A Möbius strip is made by taking a long strip of paper, giving one end a half twist, and taping the two ends together. Unlike a normal (untwisted) loop of paper, a Möbius strip has only *one* side and only *one* edge: if you trace your finger along a side, or along the edge, you will return to your starting point only after going around the strip *twice*.

A convenient way to picture this precisely is to draw a square and mark which edges get glued to which, using arrows to record the direction of gluing. Figure 2 shows this for the Möbius strip and the Klein bottle.



(a) Möbius strip: the red edges are identified with a twist. The dashed edges are not identified; together they form the single boundary loop.



(b) Klein bottle: the red edges are identified without a twist (as with a cylinder), while the blue edges are identified with a twist (as with a Möbius strip).

Figure 2. Squares with edges identified according to the matching letters and arrow directions. Same-direction arrows glue straight across; opposite-direction arrows glue with a half twist.

For the **Möbius strip** (left), the two red edges are glued together, but notice the arrows point in *opposite* directions along the square. This means the strip is glued with a half twist, exactly as with a real strip of paper. The dashed top and bottom edges are not glued to each other here; instead, they join end to end to form the strip's single boundary edge. For the **Klein bottle** (right), the red edges are glued the same way as for a cylinder (arrows point in the *same* direction, so there is no twist), while the blue edges are glued with a half twist (arrows point in *opposite* directions), just like the Möbius strip.

Figure 3. Bottom: a strip of one half of the Klein bottle before it has been stitched into a Möbius loop. **Top left**: two halves of the Klein bottle which are Möbius loops. **Top middle and top right**: stitched together Möbius loops creating a Klein bottle.

Conversely, if you take two Möbius strips and sew their single boundary loops together, with the twist reversed between the two strips, the result is a Klein bottle. This is exactly the construction used in the crochet instructions below.

3.2. **Steps requiring more explanation.**

(1) **"make slit" row 9**: In this step we continue to decrease the number of stitches by one, but also create a slit (or notch) to allow the neck of the Klein bottle to pass through the body.

For this we slip 3 stitches. Then chain 4, turn and make 3 single crochet stitches into the chain starting from the second stitch. Then single crochet one, decrease single crochet, then single crochet 4 more (to the end or the row).

3.3. **Full instructions.**

(1) Construct two strips using the instructions from Table 2, preferably with different colors. After each row, chain one, turn the strip around, and begin the new row.

Row	Instructions	Number of Stitches
1	chain 6	6
2	inc, sc, inc, sc, sc, inc	9
3	inc, 3×sc, inc, 3×sc, inc	12
4	inc, 5×sc, inc, 4×sc, inc	15
5	15×sc	15
6	dcr, 11×sc, dcr	13
7	5×sc, dcr, 6×sc	12
8	dcr, 8×sc, dcr	10
9	make slit	9
10	dcr, 5×sc, dcr	7
11	2×sc, dcr, 3×sc	6
12	dcr, sc, sc, dcr	4
13–26	4×sc	4
27	inc, 2×sc, inc	6
28–29	6×sc	6

Table 2. Row by row instructions. The first column is the row number. The second column is the instructions (refer to *glossary of abbreviations* §2.1 and *steps requiring more explanation* §3.2 for more details). The third column is the number of stitches in the row *after* completing the instructions in the second column.

(2) Take the narrow end of one strip, twist it 180 degrees, and stitch it to the base constructed in the first row.
(3) Take the other strip, and do the same but twist it in the opposite direction. You should now have two Möbius strips whose half-twists run in opposite directions.
(4) Stitch the two Möbius strips together, starting at the slits, and working away. At some point, you will need to start a new stitch and turn the strips inside out.
(5) Before finishing stitching them together, add stuffing. Some finessing and reshaping may be required to make it look like a presentable Klein bottle.

4. Further Reading

If you enjoyed this paper and would like to learn more about the Klein bottle and the topology behind it, here are some suggestions, roughly ordered from most to least accessible.

- For an introduction that assumes no calculus, read Jeffrey Weeks' book *The Shape of Space* [Wee02]. It is written for a general audience, and explains

the Klein bottle, Möbius strips, and non-orientability through hands-on paper models, much like the crochet constructions here.

- To see a Klein bottle made of glass rather than yarn, visit Cliff Stoll's website *Acme Klein Bottle* [Sto]. It is an informal and entertaining explanation of the surface by a physicist who hand-blows and sells the models himself.
- For the precise definitions behind §1, read James Munkres' textbook *Topology* [Mun00]. This is the standard undergraduate reference, and its Chapter 2 develops manifolds and orientability rigorously.
- To see the Klein bottle placed among *all* closed surfaces, read William Massey's textbook *A Basic Course in Algebraic Topology* [Mas91]. It is a graduate text, and it classifies closed surfaces using polygon edge identifications, the same technique behind Figure 2.
- For orientability, immersions, and embeddings treated with full rigor, read Manfredo do Carmo's textbook *Differential Geometry of Curves and Surfaces* [Car76]. It is a classic upper-level undergraduate text.

References

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[Wee02] J. R. Weeks. *The Shape of Space*. 2nd. CRC Press, 2002.

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